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ON AP-HENSTOCK-STIELTJES INTEGRAL OF INTERVAL-VALUED FUNCTIONS

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Abstract. In this paper we introduce the interval-valued AP-Henstock-Stieltjes integral and investigate some properties of the these integrals.

1. Introduction and preliminaries

As it is well known, the Henstock integral for a real function was first defined by Henstock [1] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Feynman integrals.

In 2000, Congxin Wu and Zengtai Gong introduced the concept of the Henstock integrals of interval-valued functions and fuzzy-number-valued functions and obtained some of its properties([7]).

In this paper we introduce the concept of the AP-Henstock-Stieltjes integral of interval-valued function and investigate some of its properties.

A Henstock partition of $[a,b]$ is a finite collection $P = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ such that $\{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a,b]$ covering $[a,b]$ and $t_i \in [c_i, d_i]$ for each $1 \leq i \leq n$. A gauge on $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$. A Henstock partition $P = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each $1 \leq i \leq n$.

Let α be an increasing function on $[a,b]$. A function $f : [a,b] \to R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a,b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such

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that $|\sum_{i=1}^{n} f(t_i)(\alpha(d_i) - \alpha(c_i)) - L| < \epsilon$ whenever $P = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a Henstock partition of $[a, b]$ subordinate to δ. We write $(H)\int_a^b f(x)d\alpha = L$ and $f \in H_\alpha[a, b]$. The function f is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$ where χ_E denotes the characteristic function of E.

Definition 1.1. Let $I_R = \{I = [I^-, I^+]$ is the closed bounded interval on the real $R\}$, where $I^- = \min\{x : x \in I\}$, $I^+ = \max\{x : x \in I\}$. For $A, B, C \in I_R$, we define $A \le B$ iff $A^- \le B^-$ and $A^+ \le B^+$, $A + B = C$ iff $A^- + B^- = C^-$ and $A^+ + B^+ = C^+$, and $AB = \{ab : a \in A, b \in B\}$, where $(AB)^- = min\{A^-B^-, A^-B^+, A^+B^-, A^+B^+\}$ and $(AB)^+ = max\{A^-B^-, A^-B^+, A^+B^-, A^+B^+\}$. Define $d(A, B) = max\{|A^- - B^-|, |A^+ - B^+|\}$ as the distance between A and B.

Definition 1.2. ([7]). Let α be an increasing function on $[a, b]$. A interval-valued function $F : [a, b] \to I_R$ is Henstock-Stieltjes integrable to $I_0 \in I_R$ with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function δ such that

$$d(\sum_{i=1}^{n} F(t_i)(\alpha(d_i) - \alpha(c_i)),\ I_0) < \epsilon$$

whenever $P = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a Henstock partition of [a,b] subordinate to δ. We write $(IH)\int_a^b F(x)d\alpha = I_0$ and $F \in IH_\alpha[a, b]$. The interval-valued function F is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$ where χ_E denotes the characteristic function of E.

2. The interval-valued AP-Henstock-Stieltjes integral

In this section, we will define the interval-valued AP-Henstock-Stieltjes integral which is an extention of the real-valued Henstock-Stieltjes integral and will study some properties of its integral.

Let E be a measurable set and let c be a real number. The density of E at c is defined by

$$d_cE = \lim_{h \to 0+} \frac{\mu(E \cap (c - h, c + h))}{2h},$$

provided the limit exists. The point c is called a point of density of E if $d_cE = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E.

An approximate neighborhood(or ad-nbd) of $x \in [a,b]$ is a measurable set $S_x \subset [a,b]$ containing x as a point of density. For every $x \in E \subset [a,b]$, choose an ad-nbd $S_x \subset [a,b]$ of x. then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([c,d],x)$ is said to fine to the choice $S = \{S_x\}$ if $c,d \in S_x$. Let $P = \{([c_i,d_i],x_i)\}_{1\le i\le n}$ be a finite collection of non-overlapping tagged intervals. If $([c_i,d_i],x_i)$ is fine to a choice S for each i, then we say that P is S -fine. Let $E \subset [a,b]$. If P is S -fine and each $x_i \in E$, then P is called S -fine on E. If P is S - fine and $[a,b] = \cup_{i=1}^n [a_i,b_i]$, then we say that P is S -fine partition of $[a,b]$.

Definition 2.1. Let α be an increasing function on $[a,b]$. A interval-valued function $F : [a,b] \to I_R$ is AP-Henstock-Stieltjes integrable to $I_0 \in I_R$ with respect to α on $[a,b]$ if for every $\epsilon > 0$ there exists a choice S on $[a,b]$ such that

$$d(\sum_{i=1}^n F(t_i)(\alpha(d_i) - \alpha(c_i)),\ I_0) < \epsilon$$

whenever $P = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a S -fine partition of $[a,b]$. We write $(APIH)\int_a^b F(x)d\alpha = I_0$ and $F \in APIH_\alpha[a,b]$. The interval-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $F\chi_E$ is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ where χ_E denotes the characteristic function of E.

Remark 2.2. It is clear, if $F(x) = F^-(x) = F^+(x)$ for all $x \in [a,b]$, then Definition 2.1 implies the real-valued AP-Henstock-Stieltjes integral.

Remark 2.3. If $F \in APIH_\alpha[a,b]$, then the integral is unique.

Theorem 2.4. *Let α be an increasing function on $[a,b]$. A interval-valued function $F : [a,b] \to I_R$ is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if $F^-, F^+ \in APH_\alpha[a,b]$ and $(APIH)\int_a^b F d\alpha = [(APH)\int_a^b F^- d\alpha, (APH)\int_a^b F^+ d\alpha]$, where $F(x) = [F^-(x), F^+(x)]$.*

Proof. Let $F \in APIH_\alpha[a,b]$. Then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for each $\epsilon > 0$ there exists a choice S on $[a,b]$ such that

$$d(\sum_{i=1}^n F(t_i)(\alpha(d_i) - \alpha(c_i)),\ I_0) < \epsilon$$

whenever $P = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is S -fine partition of $[a,b]$. Since $\alpha(d_i) - \alpha(c_i) \ge 0$ for $1 \le i \le n$, we have

$$\begin{aligned}
&d(\sum_{i=1}^{n} F(t_i)(\alpha(d_i)-\alpha(c_i)), I_0) \\
&\quad = max\left\{ |(\sum_{i=1}^{n} F(t_i)(\alpha(d_i)-\alpha(c_i))^{-} - {I_0}^{-}|, \right. \\
&\qquad\qquad \left. |(\sum_{i=1}^{n} F(t_i)(\alpha(d_i)-\alpha(c_i))^{+} - {I_0}^{+}| \right\} \\
&\quad = max\left\{ |(\sum_{i=1}^{n} F^{-}(t_i)(\alpha(d_i)-\alpha(c_i)) - {I_0}^{-}|, \right. \\
&\qquad\qquad \left. |(\sum_{i=1}^{n} F^{+}(t_i)(\alpha(d_i)-\alpha(c_i)) - {I_0}^{+}| \right\}.
\end{aligned}$$

Hence $|(\sum_{i=1}^{n} F^{-}(t_i)(\alpha(d_i)-\alpha(c_i)) - {I_0}^{-}| < \epsilon$, $|(\sum_{i=1}^{n} F^{+}(t_i)(\alpha(d_i)-\alpha(c_i)) - {I_0}^{+}| < \epsilon$ whenever $P = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is S -fine partition of $[a, b]$. Thus $F^{-}, F^{+} \in APH_{\alpha}[a, b]$ and $(APIH)\int_a^b F d\alpha = [(APH)\int_a^b F^{-} d\alpha,\ (APH)\int_a^b F^{+} d\alpha]$.

Conversely, let $F^{-}, F^{+} \in APH_{\alpha}[a, b]$. Then there exists $H_1, H_2 \in R$ with the property that given $\epsilon > 0$ there exists a choice S on [a,b] such that

$$|\sum_{i=1}^{n} F^{-}(t_i)(\alpha(d_i)-\alpha(c_i)) - H_1| < \epsilon, |\sum_{i=1}^{n} F^{+}(t_i)(\alpha(d_i)-\alpha(c_i)) - H_2| < \epsilon$$

whenever $P = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is S -fine partition on $[a, b]$. We define $I_0 = [H_1, H_2]$, then if $P = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a S - fine partitoin of $[a, b]$, we have

$$d(\sum_{i=1}^{n} F(t_i)(\alpha(d_i)-\alpha(c_i)),\ I_0) < \epsilon.$$

Hence $F : [a, b] \to I_R$ is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$. $\square$

From Theorem 2.4 and the properties of AP-Henstock-Stieltjes integral, we can easily obtain the following theorems .

THEOREM 2.5. *Let $F, G \in APIH_{\alpha}[a, b]$ and $\beta, \gamma \in R$. Then*

(1) $\beta F + \gamma G \in APIH_\alpha[a, b]$*and*

$$(APIH)\int_a^b (\beta F + \gamma G)d\alpha = \beta(APIH)\int_a^b F d\alpha + \gamma(APIH)\int_a^b G d\alpha.$$

(2) *If* $F(x) \leq G(x)$ a.e. *in* $[a, b]$, *then* $(APIH)\int_a^b F d\alpha \leq (APIH)\int_a^b G d\alpha$.

Theorem 2.6. *Let* $F \in APIH_\alpha[a, c]$ *and* $F \in APIH_\alpha[c, b]$. *Then* $F \in APIH_\alpha[a, b]$ *and* $\int_a^b F d\alpha = \int_a^c F d\alpha + \int_c^b F d\alpha$.

Theorem 2.7. *Let* $F, G \in APIH_\alpha[a, b]$ *and* $d(F, G)$ *is Lebesgue-Stieltjes integrable on* $[a, b]$. *Then*

$$d((APIH)\int_a^b F\ d\alpha, (APIH)\int_a^b G\ d\alpha) \leq (L)\int_a^b d(F, G)\ d\alpha.$$

Proof. By definition of distance,

$$d((APIH)\int_a^b F d\alpha,\ (APIH)\int_a^b G d\alpha)$$

$$= max(|((APH)\int_a^b F d\alpha)^- - ((APH)\int_a^b G d\alpha)^-|,$$

$$|((APH)\int_a^b F d\alpha)^+ - ((APH)\int_a^b G d\alpha)^+|)$$

$$= max(|(APH)\int_a^b (F^- - G^-)d\alpha|,\ |(APH)\int_a^b (F^+ - G^+)d\alpha|)$$

$$\leq max((L)\int_a^b |F^- - G^-|d\alpha,\ (L)\int_a^b |F^+ - G^+|d\alpha)$$

$$\leq (L)\int_a^b d(F, G)d\alpha.$$

□

3. The AP-Henstock-Stieltjes integrals of fuzzy-number-valued functions

Definition 3.1. ([5]). Let $\tilde{A} \in F(R)$ be a fuzzy subset on R. If for any $\lambda \in [0, 1], A_\lambda = [A_\lambda^-, A_\lambda^+]$ and $A_1 \neq \phi$, where $A_\lambda = \{x : \tilde{A}(x) \geq \lambda\}$, then $\tilde{A}$ is called a fuzzy number.

Let $\tilde{R}$ denote the set of all fuzzy numbers .

DEFINITION 3.2. ([4],[5]). Let $\tilde{A}, \tilde{B} \in \tilde{R}$, we define $\tilde{A} \leq \tilde{B}$ iff $A_\lambda \leq B_\lambda$ for all $\lambda \in (0,1]$, $\tilde{A}+\tilde{B} = \tilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$, $\tilde{A}\cdot\tilde{B} = \tilde{D}$ iff $A_\lambda \cdot B_\lambda = D_\lambda$ for any $\lambda \in (0,1]$.

LEMMA 3.3. *([1]). If a mapping* $H : [0,1] \to I_R, \lambda \to H(\lambda) = [m_\lambda, n_\lambda]$, *satisfies* $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ *when* $\lambda_1 < \lambda_2$, *then*

$$\tilde{A} := \bigcup_{\lambda \in (0,1]} \lambda H(\lambda) \in \tilde{R}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n),$$

where $\lambda_n = [1 - \frac{1}{n+1}]\lambda$.

DEFINITION 3.4. Let α be an increasing function on $[a,b]$ and let $\tilde{F} : [a,b] \to \tilde{R}$. If the interval- valued function $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ is AP-Henstock integrable on $[a,b]$ with respect to α for any $\lambda \in (0,1]$, then we say that $\tilde{F}(x)$ is AP- Henstock integrable with respect to α on $[a,b]$ and the integrable value is defined by

$$(APFH)\int_a^b \tilde{F}(x)d\alpha := \bigcup_{\lambda \in (0,1]} \lambda (IH)\int_a^b F_\lambda(x)d\alpha$$

$$= \bigcup_{\lambda \in (0,1]} \lambda[(H)\int_a^b F_\lambda^- d\alpha, (H)\int_a^b F_\lambda^+ d\alpha].$$

For brevity, we write $\tilde{F} \in APFH_\alpha[a,b]$.

THEOREM 3.5. *$\tilde{F} \in APFH_\alpha[a,b]$, then $(APFH)\int_a^b \tilde{F}(x)d\alpha \in \tilde{R}$ and*

$$[(APFH)\int_a^b \tilde{F}(x)d\alpha]_\lambda = \bigcap_{n=1}^{\infty} (APIH)\int_a^b F_{\lambda_n}(x)d\alpha,$$

where $\lambda_n = [1 - \frac{1}{n+1}]\lambda$.

Proof. Let $H : (0,1] \to I_R$ be defined by $H(\lambda) = [(H)\int_a^b F_\lambda^-(x)d\alpha, (H)\int_a^b F_\lambda^+(x)d\alpha]$. Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are increasing and decreasing on λ, respectively, therefore, when $0 < \lambda_1 \leq \lambda_2 \leq 1$, we have $F_{\lambda_1}^-(x) \leq$

$F^-_{\lambda_2}(x), F^+_{\lambda_1}(x) \geq F^+_{\lambda_2}(x)$, on $[a,b]$. Thus from Theorem 2.5, we have

$$[(H)\int_a^b F^-_{\lambda_1}(x)d\alpha, (H)\int_a^b F^+_{\lambda_1}(x)d\alpha]$$
$$\supset [(H)\int_a^b F^-_{\lambda_2}(x)d\alpha, (H)\int_a^b F^+_{\lambda_2}(x)d\alpha].$$

Using Theorem 2.5 and Lemma 3.3 we obtain

$$(APFH)\int_a^b \tilde{F}(x)d\alpha$$
$$:= \bigcup_{\lambda\in(0,1]} \lambda[(H)\int_a^b F^-_{\lambda}(x)d\alpha, (H)\int_a^b F^+_{\lambda}(x)d\alpha] \in \tilde{R}$$

and for all $\lambda \in (0,1]$,

$$[(APFH)\int_a^b \tilde{F}(x)d\alpha]_\lambda = \bigcap_{n=1}^{\infty}(APIH)\int_a^b F_{\lambda_n}(x)d\alpha,$$

where $\lambda_n = [1 - \frac{1}{n+1}]\lambda$. □

Using Theorem 3.5 and the properties of $(APIH)$ integral, we can obtain the properties of $(APFH)$ integral. For examples, the linear, monotones and interval additive properties of $(APFH)$ integral.

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